Waiting lines

Queueing theory

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PPU426

Why waiting lines form

- Temporary imbalance between demand and capacity
- Can develop even if processing time is constant

No waiting line if both demand and service rates are constant and service rate > than demand

Affects process design, capacity planning, process performance, and ultimately, supply chain performance

Use of waiting-line theory

Applies to many service or manufacturing situations

– Relating arrival and service-system processing characteristics to output

Service is the act of processing a customer

- Hair cutting in a hair salon
- Satisfying customer complaints
- Processing production orders
- Theatergoers waiting to purchase tickets
- Trucks waiting to be unloaded at a warehouse
- Patients waiting to be examined by a physician

Use of waiting-line theory

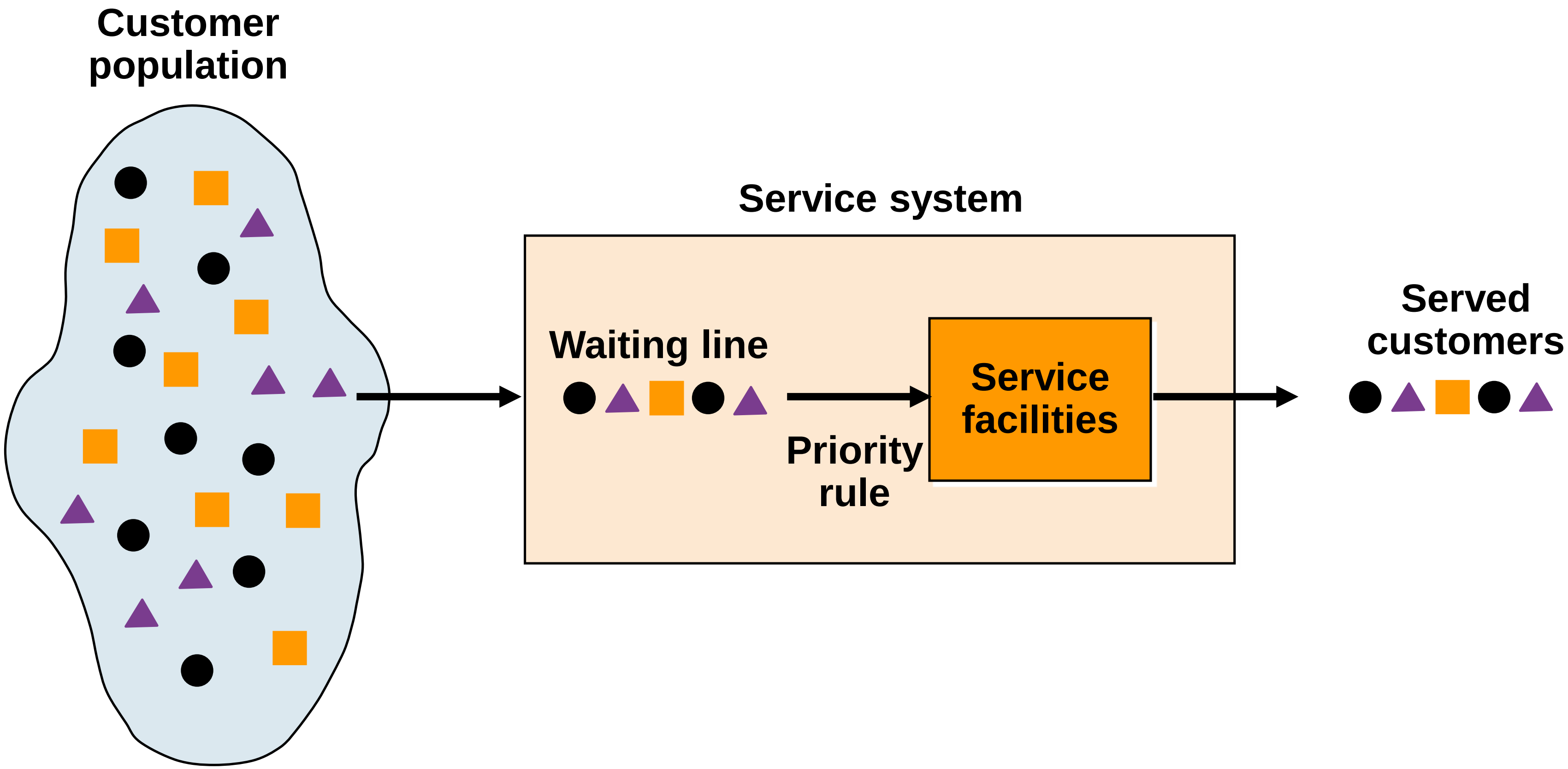
Operating characteristics

- Line length
- Number of customers in system
- Waiting time in line
- Total time in system
- Service facility utilization

Structure of waiting-line problems

1. **An input, or customer population**
2. **A waiting line of customers**
3. **The service facility**
4. **A priority rule**

Structure of waiting-line problems



Customer population

The source of input

Finite or infinite source

- Customers from a finite source reduce the chance of new arrivals
- Customers from an infinite source do not affect the probability of another arrival

Customers are *patient* or *impatient*

- Patient customers wait until served
- Impatient customer either balk or join the line and renege

The Service system

Number of lines

- A single-line keeps servers uniformly busy and levels waiting times among customers
- A multiple-line arrangement is favored when servers provide a limited set of services

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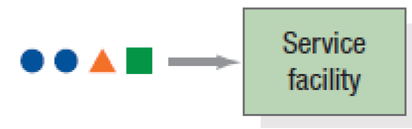
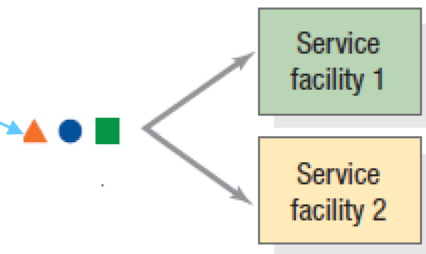
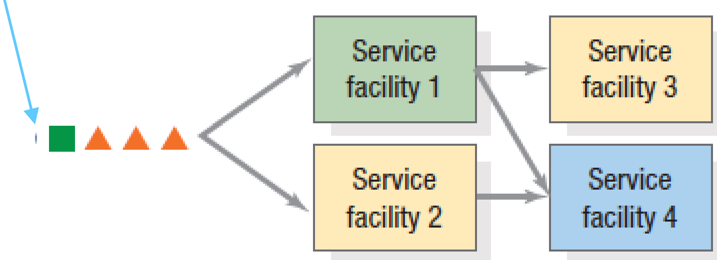
The Service system

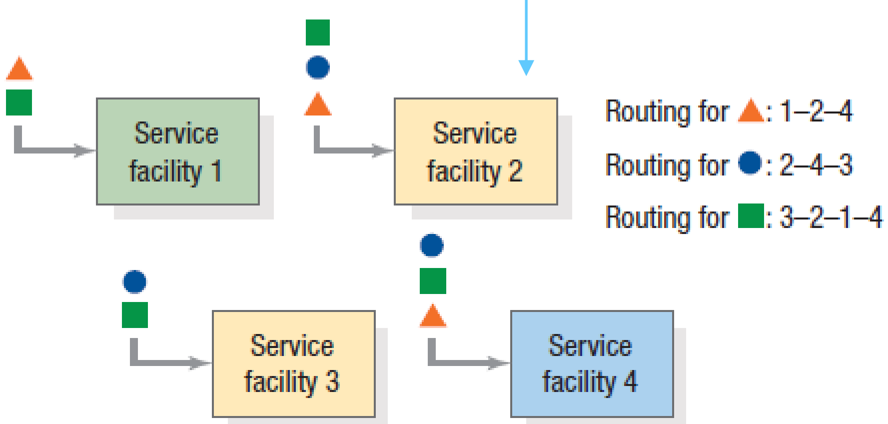
Arrangement of service facilities

- Single-channel, single-phase
- Single-channel, multiple-phase
- Multiple-channel, single-phase
- Multiple-channel, multiple-phase
- Mixed arrangement

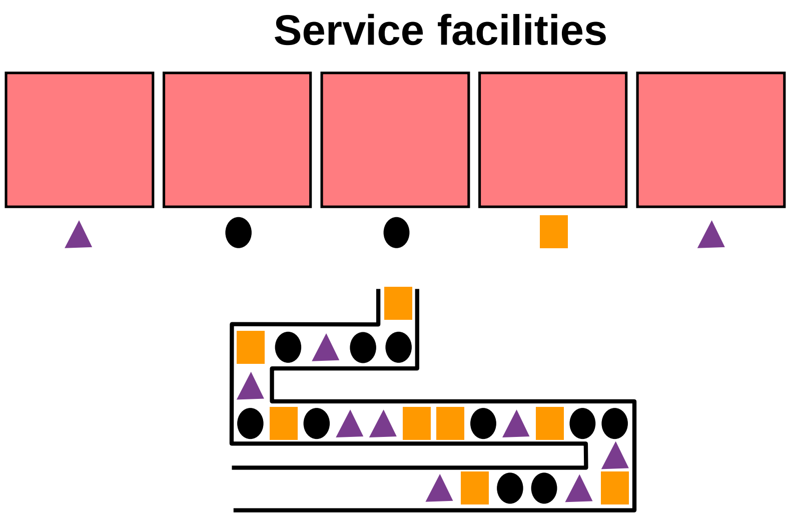




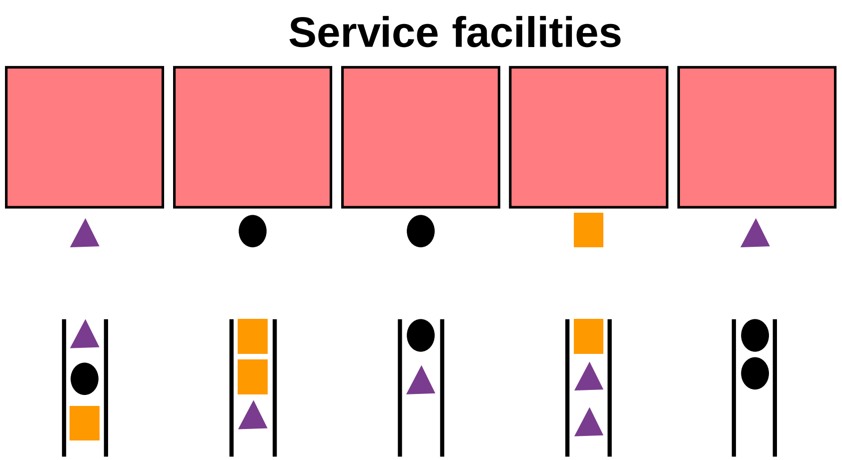




The Service system



(a) Single line



(b) Multiple lines

Priority rules

First-come, first-served (FCFS)—used by most service systems (Also known as FIFO: first-in, first-out)

Other rules, e.g.

- Last in, first out (LIFO)
- Earliest due date (EDD)
- Shortest processing time (SPT)

Probability distributions

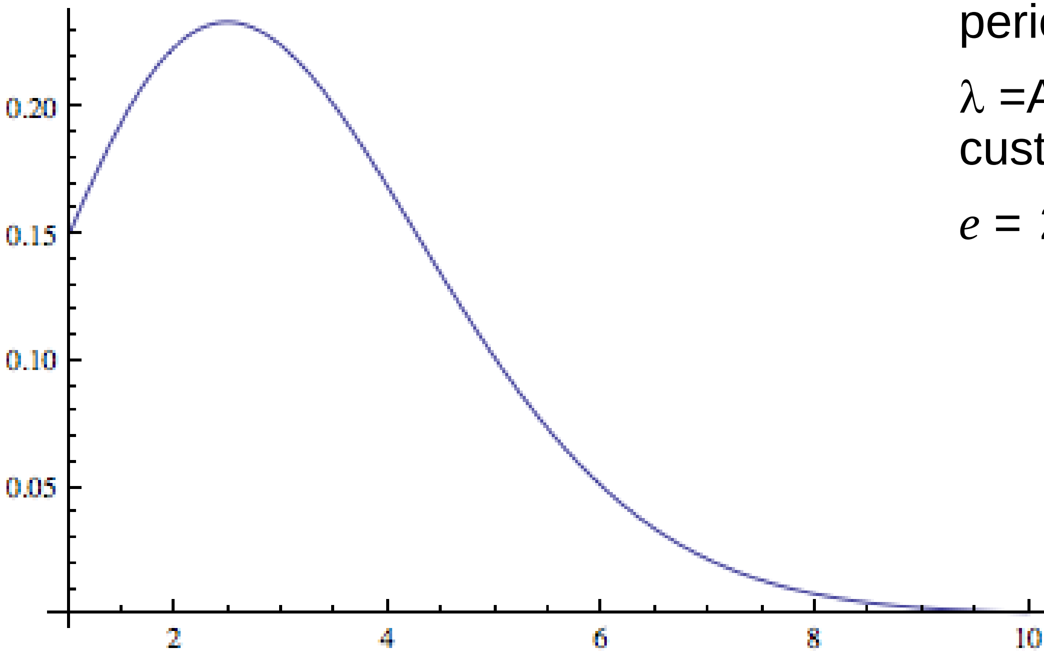
The sources of variation in waiting-line problems come from the random arrivals of customers and the variation of service times

Arrival distribution

- Customer arrivals can often be described by the Poisson distribution with mean $= \lambda T$ and variance also $= \lambda T$
- Arrival distribution is the probability of n arrivals in T time periods
- Interarrival times are the average time between arrivals

Interarrival times

$$P_n = \frac{(\lambda T)^n}{n!} e^{-\lambda T} \quad \text{for } n = 0, 1, 2, \ldots$$



Where

P_n =Probability of n arrivals in T time periods

λ =Average numbers of customer arrivals per period

e = 2.7183

Probability of customer arrivals

EXAMPLE

Management is redesigning the customer service process in a large department store. Accommodating four customers is important. Customers arrive at the desk at the rate of two customers per hour. What is the probability that four customers will arrive during any hour?

$$P_n = \frac{(\lambda T)^n}{n!} e^{-\lambda T} \quad \text{for } n = 0, 1, 2, \ldots$$

Probability of customer arrivals

EXAMPLE

Management is redesigning the customer service process in a large department store. Accommodating four customers is important. Customers arrive at the desk at the rate of two customers per hour. What is the probability that four customers will arrive during any hour?

SOLUTION

In this case customers per hour, T = 1 hour, and n = 4 customers. The probability that four customers will arrive in any hour is

$$P_4 = \frac{[2(1)]^4}{4!} e^{-2(1)} = \frac{16}{24} e^{-2} = 0.090$$

Service time

Service time distribution can be described by an exponential distribution with mean $= 1/\mu$ and variance $= (1/\mu)^2$

Service time distribution: The probability that the service time will be no more than *T* time periods can be described by the exponential distribution

$$P(t \leq T) = 1 - e^{-\mu T}$$

where

μ = average number of customers completing service per period

t = service time of the customer

T = target service time

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Service time probability

EXAMPLE

The management of the large department store in Example 1 must determine whether more training is needed for the customer service clerk. The clerk at the customer service desk can serve an average of three customers per hour. What is the probability that a customer will require less than 10 minutes of service?

$$P(t \leq T) = 1 - e^{-\mu T}$$

Service time probability

EXAMPLE

The management of the large department store must determine whether more training is needed for the customer service clerk. The clerk at the customer service desk can serve an average of *three customers per hour*. What is the probability that a customer will require *less than 10 minutes* of service?

SOLUTION

We must have all the data in the same time units. Because μ = 3 customers per hour, we convert minutes of time to hours, or T = 10 minutes = 10/60 hour = 0.167 hour. Then

$$P(t \leq T) = 1 - e^{-\mu T}$$

$$P(t \leq 0.167 \text{ hour}) = 1 - e^{-3(0.167)} = 1 - 0.61 = 0.39$$

Single server model

Single-server, single line of customers, and only one phase

Assumptions are

1. Customer population is infinite and patient
2. Customers arrive according to a Poisson distribution, with a mean arrival rate of λ
3. Service distribution is exponential with a mean service rate of μ
4. Mean service rate exceeds mean arrival rate
5. Customers are served FCFS
6. The length of the waiting line is unlimited

Single server model

Average utilization of the system: $\rho = \dfrac{\lambda}{\mu}$

Average number of customers in the service system: $L = \dfrac{\lambda}{\mu - \lambda}$

Average number of customers in the waiting line: $L_q = \rho L$

Average time spent in the system, including service: $W = \dfrac{1}{\mu - \lambda}$

Average waiting time in line: $W_q = \rho W$

Single server model

Average utilization of the system: $\rho = \dfrac{\lambda}{\mu}$

Probability that n **customers are in the system:** $P_n = (1 - \rho)\rho^n$

Probability that 0 **customers are in the system:** $P_0 = 1 - \dfrac{\lambda}{\mu}$

Probability that n **customers are in the system:** $P_n = P_0 \left(\dfrac{\lambda}{\mu}\right)^n$

Probability less than n **customers are in the system:** $P_{<n} = 1 - \left(\dfrac{\lambda}{\mu}\right)^n$

Application

EXAMPLE:

Customers arrive at a checkout counter at an average 20 per hour, according to a Poisson distribution. They are served at an average rate of 25 per hour, with exponential service times. Use the single-server model to estimate the operating characteristics of this system.

λ = 20 customer arrival rate per hour

μ = 25 customer service rate per hour

Calculate:

1. **Average utilization of system**
2. **Average number of customers in the system**
3. **Average number of customers in the waiting line**
4. **Average time spent in the system**
5. **Average waiting time in line**
6. **The probability of 0 customers in the system**
7. **The probability of more than 0, 1, 2, 3, 4, and 5 customers in the system**

Application

SOLUTION

1. **Average utilization of system** $$\rho = \frac{\lambda}{\mu} = \frac{20}{25} = 0.8$$

2. **Average number of customers in the service system** $$L = \frac{\lambda}{\mu - \lambda} = \frac{20}{25 - 20} = 4$$

3. **Average number of customers in the waiting line** $$L_q = \rho L = 0.8(4) = 3.2$$

4. **Average time spent in the system, including service** $$W = \frac{1}{\mu - \lambda} = \frac{1}{25 - 20} = 0.2$$

5. **Average waiting time in line** $$W_q = \rho W = 0.8(0.2) = 0.16$$

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Application

SOLUTION

6. Probability of 0 customers in the system

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{20}{25} = 0.2$$

7. Probability of more than 0, 1, 2, 3, 4, and 5 customers in the system

Probability of more than *k* customers in the system:

$$\boldsymbol{P_{n>k} = (\frac{\lambda}{\mu})^{k+1} = (\frac{20}{25})^{k+1}}$$

$$P_{n>0} = 0.8$$
$$P_{n>1} = 0.64$$
$$P_{n>2} = 0.51$$
$$P_{n>3} = 0.41$$
$$P_{n>4} = 0.33$$
$$P_{n>5} = 0.26$$

Relevant book chapters

- Supplement B: “Waiting line models”

Thank you!

Questions?

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